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José Manuel Giménez Gómez
Josep E. Peris
María José Solís Baltodano

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Departament d'Economia
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Universitat Rovira i Virgili
Facultat d'Economia i Empresa
Av. de la Universitat, 1
43204 Reus
Tel.: +34 977 759 811
Fax: +34 977 758 907
Email: sde@urv.cat

CREIP
www.urv.cat/creip
Universitat Rovira i Virgili
Departament d'Economia
Av. de la Universitat, 1
43204 Reus
Tel.: +34 977 758 936
Email: creip@urv.cat

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Lower bounds and resource allocation

José-Manuel Giménez-Gómez[a], Josep E. Peris[b]
and María-José Solís-Baltodano[c]

[a] *Universitat Rovira i Virgili, Departament d'Economia and CREIP, Av.Universitat 1, 43204 Reus, Spain. (josemanuel.gimenez@urv.cat)*
[b] *Universitat d'Alacant, Mètodes Quantitatius i Teoria Econòmica (MQiTE) and Institut Interuniversitari Desenvolupament Social i Pau (IUDESP), 03080 Alacant, Spain. (peris@ua.es, subiza@ua.es)*
[c] *Universitat Rovira i Virgili, Departament d'Economia and CREIP, Av.Universitat 1, 43204 Reus, Spain. (mariajose.solis@urv.cat)*

Abstract

The existence of lower bounds that guarantee a minimum to each agent involved in an allocation (claims) problem has been widely analysed in the literature. When focusing in claims problems, four lower bounds in what any individual should receive are the *minimal right* (Curiel et al., 1987), the *fair lower bound* (Moulin, 2002), *securement* (Moreno-Ternero and Villar, 2004) and the *min lower bound* Dominguez (2013). In the current paper we impose that a mechanism should fulfill one of the mentioned lower bound (we call this property *respect of the lower bound*, RB) and compare the mechanisms obtained by using RB together with some additional properties. We provide new characterizations of the so-called uniform rules (constrained equal awards, constrained equal losses) and the Ibn Ezra's proposal.

Keywords: claims problems; lower bounds; claims rules

1. Introduction

The so-called claims problem reflects a situation where the aggregate claim of a group of individuals is greater than the resources to be distributed among them. The way of rationing the endowment among the agents, taking into account their claims, is prescribed by a rule. In the current paper we analyze how to distribute any increment of the endowment in terms of two general concepts: equal treatment and warranty (which is determined by a lower bound on what an agent receives).

The concern of ensuring some minimum individual rights has figured in a large number of contexts. Specifically, the *Universal Basic Income* is a classical issue that has attracted most of the attention in the social policy literature and the political agenda during the last two decades (Noguera, 2010). The establishment of a minimum wage in the labour market, or the debate of ensuring a universal minimum health coverage in the U.S. Senate, are another examples. Furthermore, the idea of establishing warranties underlies the theoretical analysis of claims problems from its beginning (O'Neill, 1982) to present day (Giménez-Gómez and Marco-Gil, 2014).

In Dominguez (2013) the recursive application of a lower bound is analysed and some claims rules are characterized by using some additional property. Instead of applying recursively the lower bound, we only ask that the proposed solution fulfills this bound, that is it assigns to each individual at least the amount determined by the lower bound.

Moreover, we impose some additional requirement that depends on the lower bound being used: *conditional equal treatment*, *conditional resource monotonicity*, *conditional group solidarity*, or *priority*.

Then, a crucial point in our study is the selection of a particular lower bound with respect to which the above mentioned properties are applied. Since we are interested in comparing lower bounds among agents, we need to choose a *significant* one, in the sense that it should be different from zero, whenever the claim is. We analyse four lower bounds defined in the literature: the *minimal right* (Curiel et al., 1987), the *fair lower bound* (Moulin, 2002), *securement* (Moreno-Ternero and Villar, 2004) and the *min lower bound* Dominguez (2013). The impact of requiring that s claims rule fulfills a lower bound is analysed in Dominguez and Thomson (2006) and Yeh (2008).

The remainder of the paper is organized as follows. Next section presents the model and introduces the lower bounds. Section 4 introduces the axioms and provides our main results. Finally, Section 5 closes the paper.

2. Preliminaries

Throughout the paper we will consider a set of agents $N = \{1, 2, ..., n\}$. Each agent, $i \in N$, is identified by her *claim*, c_i, on some *endowment* $E > 0$. The *aggregate claim*, C, is given by $C = \sum_{i=1}^{n} c_i$. A *claims problem* appears whenever the endowment is not enough to satisfy the aggregate claim, that is, $C > E$. Without loss of generality, we assume that the agents are indexed according to their claims, $c_1 \leqslant c_2 \leqslant \ldots \leqslant c_n$. The pair $(E, c) \in \mathbb{R}_{++} \times \mathbb{R}^n_+$ represents the claims problem, and $\mathcal{B}$ denotes the set of all claims problems. A *rule* is a single valued function $\varphi : \mathcal{B} \to \mathbb{R}^n_+$ such that for each $(E, c) \in \mathcal{B}$, and each $i \in N$, fulfills $0 \leqslant \varphi_i(E,c) \leqslant c_i$ (*non-negative* and *claim-bounded*), and $\sum_{i=1}^{n} \varphi_i(E,c) = E$ (*efficient*).

Two of the most important rules proposed by the literature are the uniform ones (Maimonides, 12th century): the *constrained equal awards* rule (that recommends an equal distribution of the endowment subject to that no one receives more than her claim) and the the *constrained equal losses* rule (that recommends an equal loss of the claim subject to that no one receives a negative amount). Another classical rule (analysed in Alcalde et al. (2005)) is the *Ibn Ezra* (attributed to Rabbi Abraham Ibn Ezra in the 12th century): this rule suggest that each individually claimed unit should be divided equally among all agents claiming it. Formally, this rules can be defined as follows.

The **constrained equal awards** rule, *cea*:

For each (E,c) in $\mathcal{B}$ and each $i \in N$, $cea_i(E,c) \equiv \min\{c_i, \mu\}$, where μ is chosen so that $\sum_{i \in N} \min\{c_i, \mu\} = E$.

The **constrained equal losses** rule, *cel*:

For each (E,c) in $\mathcal{B}$ and each $i \in N$, $cel_i(E,c) \equiv \max\{0, c_i - \mu\}$, where μ is chosen so that $\sum_{i \in N} \max\{0, c_i - \mu\} = E$.

The **Ibn Ezra** rule, IE:

For each (E,c) in $\mathcal{B}$, such that $c_n \leqslant E$, $c_i \leqslant c_{i+1}$, and each $i \in N$,

$$\varphi_i^{IE}(E,c) \equiv \sum_{k=1}^{i} \frac{\min(c_k, E) - \min(c_{k-1}, E)}{n-k+1},$$

where, for notational convenience, we will consider $c_0 = 0$.

Ibn Ezras recommendation can be understood as follows (see Alcalde et al. (2005)): Let us consider that from the total amount to share $[0, E]$, each agent i demands the specific parts of the state $[0, c_i]$; once the claims are arranged on specific units of the estate in this way, Ibn Ezra recommends for each unit equal division among all agents demanding it. Ibn Ezra gives an example where an state $E = 120$ must be divided between four agents with claims $30, 40, 60, 120$. The claims problem is $(E,c) = (120, (30, 40, 60, 120))$ and the IE rule proposes the following assignment: each agent receives $\frac{c_1}{n}$; the second and successive agents additionally receive $\frac{c_2 - c_1}{n-1}$; the third and fourth agents will additionally receive $\frac{c_3 - c_2}{n-2}$, and the last agent will additionally receive the remaining. In the numerical example that gives the following sharing of the endowment:

$IE(E,c) = \left(\frac{30}{4}, \frac{30}{4} + \frac{10}{3}, \frac{30}{4} + \frac{10}{3} + \frac{20}{2}, \frac{30}{4} + \frac{10}{3} + \frac{20}{2} + 60\right).$

In order to compare the three solutions we have introduced, we obtain the result of applying the *cea* and *cel* rules to this numerical example.

$cea(E,c) = (30, 30, 30, 30)$ $\qquad$ $cel(E,c) = \left(0, \frac{20}{3}, \frac{80}{3}, \frac{260}{3}\right).$

To conclude this section, it is noteworthy that the concept of lower bound has been always present as a crucial point in claims problems. Indeed, the idea of pretending to ensure to each agent a minimal amount already appears in the formal definition of a rule, by the non-negativity condition. In general, a lower bound is a function such that for each claims problem (E, c) and each agent $i \in N$, $b_i(E, c)$ is the minimal amount that agent i should receive in this claims situation according to b. This lower bounds should fulfill two conditions:

(i) Is *rational*: the guaranteed minimum is non-negative and lower that the agent's claim.
(ii) Is *feasible*: the estate allows to assign these amounts to the agents.

A general formal definition is given in Dominguez (2013).

A lower bound is a function, $b : \mathcal{B} \to \mathbb{R}^n_+$ which maps each problem $(E, c) \in \mathcal{B}$, and each $i \in N$, to a real number $b_i(E, c)$ such that

(i) $0 \leqslant b_i(E, c) \leqslant c_i$
(ii) $\sum_{i=1}^{n} b_i(E, c) \leqslant E$

Curiel et al. (1987) introduced a lower bound, which is called *minimal right*, which requires that each agent receives what is available whenever the other agents have already received their claim, or zero if this is not possible. Moulin (2002) introduces a lower bound, the *fair lower bound*, which establishes that all agents should receive at least the amount assigned to each of them in a equal division, or their full claim. Moreno-Ternero and Villar (2004) propose a lower bound they call *securement*, that guarantees (if possible) the $n-th$ part of each agent's claim (in other case, it guarantees an equal division of the endowment). Finally, a recent lower bound, *min lower bound*, introduced in Dominguez (2013), proposes that each agent receives (if possible) the $n-th$ part of the smallest claim (in other case, it guarantees an equal division of the endowment). Next, we introduce formally these lower bounds.

Minimal right, (Curiel et al., 1987): for each $(E, c) \in \mathcal{B}$ an each $i \in N$,

$$mr_i(E, c) = \max \left\{ 0, E - \sum_{j \in N \setminus \{i\}} c_j \right\}.$$

Fair lower bound, (Moulin, 2002): for each $(E, c) \in \mathcal{B}$ and each $i \in N$,
$f_i^l(E, c) = \min \left\{ c_i, \frac{E}{n} \right\}.$

Securement, (Moreno-Ternero and Villar, 2004): for each $(E, c) \in \mathcal{B}$ and each $i \in N$,
$S_i(E, c) = \frac{1}{n} \min \{c_i, E\}.$

Min lower bound, (Dominguez, 2013): for each $(E, c) \in \mathcal{B}$ and each $i \in N$,
$m_i^l(E, c) = \frac{1}{n} \min \left\{ \min c_{j_{j \in N}}, E \right\}.$

3. Axioms

Next we introduce some axioms, which are referred to some lower bound, in order to analyze the claims rules satisfying such properties. The first one is our basic requirement: the lower bound is respected by the claims rule.

Axiom RB. Respect of a lower bound b: for each $(E, c) \in \mathcal{B}$, and all $i \in N$, $\varphi_i(E, c) \geqslant b_i(E, c)$.

RB requires that each agent should receive at least her lower bound (so each agent has a guaranteed minimum level of awards).

Axiom ETEB. Conditional equal treatment with respect to a lower bound b: for each $(E, c) \in \mathcal{B}$, and all $i, j \in N$, $c_i \leqslant c_j$, $b_i(E, c) = b_j(E, c)$ implies $\varphi_i(E, c) = \varphi_j(E, c)$, or $\varphi_i(E, c) = c_i \leqslant \varphi_j(E, c)$.

ETEB states equal treatment for equal agents (regarding their lower bounds), unless one of them has her demand met in full. That implies anonymity with respect to the claims. Note that the second part of the axiom is required since by asking for equal treatment with respect to the lower bound b_i, it may lead to giving an individual more than her claim, which is not possible in a claims rule.

Axiom CRM. Conditional resource monotonicity with respect to a lower bound b: if $(E, c), (E', c) \in \mathcal{B}$ are such that $E > E'$, then for all $i \in N$, $\varphi_i(E, c) - \varphi_i(E', c) \geqslant b_i(E, c) - b_i(E', c)$, or $\varphi_i(E, c) = c_i$.

CRM, asks for a stronger condition: *any change in the awards received by each individual due to a change in the endowment* E *should be at least equal to the change in her bound*. As before, we need to restrict this idea in order that no individual receives more than her claim.

Axiom CGS. Conditional group solidarity for equal changes in a lower bound b**:** if $(E, c), (E', c) \in \mathcal{B}$ are such that $E > E'$, then for all $i, j \in N$, $c_i \leqslant c_j$, $b_i(E, c) - b_i(E', c) = b_j(E, c) - b_j(E', c)$ implies,

$$\varphi_i(E, c) - \varphi_i(E', c) = \varphi_j(E, c) - \varphi_j(E', c), \text{ or } \varphi_i(E, c) = c_i \leqslant \varphi_j(E, c).$$

CGS requires that if the endowment increases, then this increment should be shared equally among agents who experiment an equal change in their lower bound. As before, this increment needs to be limited by the claim of each individual.

Axiom PRI. Priority b**:** if $(E, c), (E', c) \in \mathcal{B}$ are such that $E > E'$, then only for each $i \in N$ such that $b_i(E, c) - b_i(E', c) > 0$, $\varphi_i(E, c) - \varphi_i(E', c) > 0$.

PRI states that only those agents who increases their lower bound, should increase their allocation.

4. Main results

In this section we provide some characterizations of the *constrained equal awards* and *Ibn Ezra* rules. We analyze the effect of each lower bound.

4.1. Fair bound

Our first result combines the property of *Respect of a lower bound*, with *Conditional equal treatment with respect to a lower bound.* Theorem 1 shows that with the fair lower bound, by requiring RB and $ETEB$, we retrieve the *cea* rule.

Theorem 1. *Let us consider the bound* $b = f^l$*. Then, the cea rule is the only one satisfying RB and ETEB.*

Proof. It is clear that *cea* satisfies RB. In order to prove that it fulfills $ETEB$, let us consider $i \leqslant j$ such that $f_i^l(E, c) = f_j^l(E, c)$. If $f_i^l(E, c) = c_i$, then

$cea_i(E,c) = cea_j(E,c)$, unless $cea_i(E,c) = c_i \leqslant cea_j(E,c)$. Furthermore, if $f_i^l(E,c) = \frac{E}{n}$: then $cea_i(E,c) = cea_j(E,c)$. So, the *cea* rule satisfies *ETEB*.

Now, for each $(E,c) \in \mathcal{B}$ and each $i \in N$, we are going to prove that a rule φ satisfying axioms *RB* and *ETEB* coincides with the *cea* rule.

As $E < \sum_{i=1}^n c_i < nc_n$, there is some $k \in N$ such that $E < nc_k$. If $E < nc_1$, then $f_i^l(E,c) = \frac{E}{n}$ for all $i \in N$ and *RB* implies $\varphi_i(E,c) = \frac{E}{n} = cea_i(E,c)$ for all $i \in N$, so we obtain in this case that $\varphi = cea$.

In other case, there is some $k \in N$ such that $nc_{k-1} \leqslant E < nc_k$. For all $i \leqslant k-1$, $f_i^l(E,c) = c_i$, and for all $i > k-1$, $f_i^l(E,c) = \frac{E}{n}$. By *RB* and *claim-boundedness*, for all $i \leqslant k-1$, $\varphi_i(E,c) = c_i$, and *ETEB* and *efficiency* will imply an equal sharing of $E' = E - (c_1 + c_2 + \ldots + c_{k-1})$, among agents $i = k, \ldots, n$, unless some of them gets more than her claim.

If $\frac{E'}{n-(k-1)} > c_k$, then *ETEB* and *claim-boundedness* implies $\varphi_k(E,c) = c_k$. Now, by *ETEB*, $\varphi_i(E,c) = \varphi_j(E,c)$, for all $i,j > k$, and *efficiency* implies $\varphi_i(E,c) = \frac{E-\sum_{i=1}^k c_i}{n-k}$ for all $i > k$, unless some of these amounts are greater than the respective claims.

If $\frac{E''}{n-k} > c_{k+1}$, $E'' = E-(c_1+c_2+\ldots+c_k)$, *ETEB* and *claim-boundedness* implies $\varphi_{k+1}(E,c) = c_{k+1}$ and the remainder must be distributed equally by *ETEB*. This argument is repeated until no one gets more than her claim, and we observe that the result is $\varphi(E,c) = cea(E,c)$. ■

Theorem 2 establishes that, if we use the fair lower bound, *RB* and *ETEB* characterize the *cea* rule.

Theorem 2. *Let us consider the bound $b = f^l$. Then, the cea rule is the only one satisfying ETEB and CRM.*

Proof. Let φ a rule satisfying axioms *ETEB* and *CRM*. If we apply *CRM* to problems (E,c) and (E',c), with $E' = 0$, then we obtain that *RB* is fulfilled. So by applying the result in Theorem 1 we obtain that φ coincides with *cea*. Now, we need to prove that the *cea* rule satisfies both axioms. It has been proved that *ETEB* is satisfied. In order to prove that it fulfills *CRM*, let us consider $E' < E$ and $i \in N$. If $cea_i(E,c) < c_i$, then $cea_i(E',c) < c_i$ since *cea* satisfies *resource monotonicity*. In this case,

$$cea_i(E,c) - cea_i(E',c) =$$

$$= \left(\frac{E}{n} + cea_i(F, c - f_i^l(E,c))\right) - \left(\frac{E'}{n} + cea_i(F', c - f_i^l(E',c))\right),$$

where $F = E - \sum_{i=1}^n f_i^l(E,c); \quad F' = E' - \sum_{i=1}^n f_i^l(E',c)$. So,

$$cea_i(E,c) - cea_i(E',c) \geqslant \frac{E}{n} - \frac{E'}{n} = f_i^l(E,c) - f_i^l(E',c).$$

If $cea_i(E,c) = c_i$, the property is then fulfilled. ■

If we now combine the *RB* and *CGS*, again the *cea* rule is characterized.

Theorem 3. *Let us consider the bound $b = f^l$. Then, the cea rule is the only one satisfying RB and CGS.*

Proof. Let φ a rule satisfying axioms *RB* and *CGS*. If we apply *CGS* to problems (E,c) and (E',c), with $E' = 0$, then we obtain that *ETEB* is fulfilled. So by applying the result in Theorem 1 we obtain that φ coincides with *cea*. Now, in order to prove that the *cea* rule fulfills axiom *CGS*, let us consider i, j such that for some $E > E'$, $f_i^l(E,c) - f_i^l(E',c) = f_j^l(E,c) - f_j^l(E',c), c_i \leqslant c_j$. If $f_i^l(e,c) = \frac{E}{n}$, then $f_i^l(E',c) = \frac{E'}{n}$. In this case, either (i) $cea_i(E,c) = cea_j(E,c)$, so $cea_i(E',c) = cea_j(E',c)$; or (ii) $cea_i(E,c) = c_i \leqslant cea_j(E,c)$. Furthermore, if $f_i^l(E,c) = c_i$, $cea_i(E,c) = c_i \leqslant cea_j(E,c)$. Hence, the *cea* rule satisfies *CGS*. ■

As we have shown, *CRM* implies *RB* and *CGS* implies *ETEB*. So, by combining *CRM* and *CGS* we obtain a new characterization result.

Theorem 4. *Let us consider the bound $b = f^l$. Then, the cea rule is the only one satisfying CRM and CGS.*

4.2. Min lower bound

Next, Theorem 5 shows that by using the *Min lower bound*, requiring *RB* and *ETEB* also characterizes the *cea* rule.

Theorem 5. *Let us consider the bound $b = m^l$. Then, the cea rule is the only one satisfying RB and ETEB.*

Proof. It is clear that *cea* satisfies *RB*. In order to prove that it fulfills *ETEB*, let us consider $i \leqslant j$ such that $m^l_i(E,c) = m^l_j(E,c)$. If $m^l_i(E,c) = \frac{E}{n}$, then $E \leqslant c_1$. Hence, $cea_i(E,c) = cea_j(E,c) = \frac{E}{n}$. If $m^l_i(E,c) = \frac{c_1}{n}$, then $cea_i(E,c) = cea_j(E,c) \leqslant \frac{E}{n}$, unless $cea_i(E,c) = c_i \leqslant cea_j(E,c)$. So, the *cea* rule satisfies *ETEB*.

Now, for each $(E,c) \in \mathcal{B}$ and each $i \in N$, we are going to prove that a rule φ satisfying axioms *RB* and *ETEB* coincides with the *cea* rule.

If $E \leqslant c_1$, then $m^l_i(E,c) = \frac{E}{n}$ for each $i \in N$. By *RB*, for each $i \in N$, $\varphi_i(E,c) \geqslant \frac{E}{n}$. Thus, by efficiency, $\varphi_i(E,c) = \frac{E}{n} = cea_i(E,c)$, since $\frac{E}{n} \leqslant c_i$ for all i.

If $c_1 < E < \sum\limits_{i=1}^{n} c_i$, then $m^l_i(E,c) = \frac{c_1}{n}$, for each $i \in N$. By *RB*, for each $i \in N$, $\varphi_i(E,c) \geqslant \frac{c_1}{n}$. By $ETEB$, for each $i,j \in N$ such that $m^l_i(E,c) = m^l_j(E,c)$, $\varphi_i(E,c) = \varphi_j(E,c)$ or $\varphi_i(E,c) = c_i \leqslant \varphi_j$. Then claim-boundedness by efficiency reply $\varphi_i = c_i$ or $\varphi_i = \mu$, such that $\sum \varphi_i = E$ that is, $\varphi = cea$.

If $E \leqslant nc_1$, then $\varphi_i(E,c) = \frac{E}{n} = cea_i(E,c)$, for each $i \in N$.

If $nc_1 < E \leqslant (n-1)c_2 + c_1$, by *claim-boundedness*, $\varphi_1(E,c) = c_1$. By *ETEB* and *efficiency*, $\varphi_j(E,c) = \frac{E-c_1}{n-1}$ for each $j > 1 \in N$. So, $\varphi(E,c) = cea(E,c)$.

Hence, for some $k \in N$, if $(n-(k-1))c_k + \sum\limits_{i=1}^{k-1} c_i < E \leqslant (n-(k-2))c_{k-1} + \sum\limits_{i=1}^{k-2} c_i$, by *claim-boundedness*, $\varphi_i(E,c) = c_i$, for each $i \leqslant k-1 \in N$. By *ETEB* and *efficiency*, $\varphi_j(E,c) = \frac{E - \sum\limits_{i=1}^{k-1} c_i}{n-(k-1)}$ for each $j \geqslant k \in N$. So, $\varphi(E,c) = cea(E,c)$. ■

The last result remains valid if we use the $ETEB$ and CRM, as Theorem 6 shows.

Theorem 6. *Let us consider the bound $b = m^l$. Then, the cea rule is the only one satisfying ETEB and CRM.*

Proof. Let φ a rule satisfying axioms *ETEB* and *CRM*. If we apply *CRM* to problems (E,c) and (E',c), with $E' = 0$, then we obtain that *RB* is fulfilled. So by applying the result in Theorem 4 we obtain that φ coincides with *cea*. Now, we need to prove that the *cea* rule satisfies both axioms. It has been proved that *ETEB* is satisfied. In order to prove that it fulfils *CRM*, let us consider $E' < E$ and $i \in N$. If $cea_i(E,c) \leqslant c_i$, then $cea_i(E',c) < c_i$ since *cea* satisfies *resource monotonicity*. In this case,

$$cea_i(E,c) - cea_i(E',c) =$$

$$= \left(\frac{E}{n} + cea_i(F, c - m^l(E,c))\right) - \left(\frac{E'}{n} + cea_i(F', c - m^l(E',c))\right),$$

where $F = E - \sum_{i=1}^{n} m_i^l(E,c)$; $\quad F' = E' - \sum_{i=1}^{n} m_i^l(E',c)$. So,

$$cea_i(E,c) - cea_i(E',c) \geqslant \frac{E}{n} - \frac{E'}{n} = m_i^l(E,c) - m_i^l(E',c).$$

If $cea_i(E,c) = c_i$, the property is then fulfilled. ■

Theorem 7. *Let us consider the bound $b = m^l$. Then, the cea rule is the only one satisfying RB and CGS.*

Proof. Let φ a rule satisfying axioms *RB* and *CGS*. If we apply *CGS* to problems (E,c) and (E',c), with $E' = 0$, then we obtain that *ETEB* is fulfilled. So by applying the result in Theorem 4 we obtain that φ coincides with *cea*. Now, in order to prove that the *cea* rule fulfills axiom *CGS*, let us consider i,j such that for some $E > E'$, $m_i^l(E,c) - m_i^l(E',c) = m_j^l(E,c) - m_j^l(E',c), c_i \leqslant c_j$. If $m_i^l(E,c) = \frac{E}{n}$, then $m_i^l(E',c) = \frac{E'}{n}$. In this case, either (i) $cea_i(E,c) = cea_j(E,c)$, so $cea_i(E',c) = cea_j(E',c)$; or (ii) $cea_i(E,c) = c_i \leqslant cea_j(E,c)$. Furthermore, if $minb_i^l(E,c) = c_i$, $cea_i(E,c) = c_i \leqslant cea_j(E,c)$. Hence, the *cea* rule satisfies *CGS*. ■

We have shown that *CRM* implies *RB* and that *CGS* implies *ETEB*. So, by combining *CRM* and *CGS* we obtain a new characterization result.

Theorem 8. *Let us consider the bound $b = m^l$. Then, the cea rule is the only one satisfying CRM and CGS.*

4.3. Securement

From the results in the previous sub-sections we may ask if for every lower bound b, the introduced axioms characterize the *cea* rule. We prove that this is not true, since by using the *securement lower bound* Theorem 9 shows that requiring *RB*, *CGS* and *PRI* retrieves the Ibn Ezra proposal.

Theorem 9. *Let us consider the bound $b = S^l$. Then, the IE rule is the only one satisfying RB, CGS and PRI.*

Proof. In order to prove that IE satisfies RB, let us consider that for each $i \in N$, $IE_i(E,c) \geqslant S^l_i(E,c) = \frac{1}{n}min\{c_i, E\}$. In doing so, $IE_i(E,c) \geqslant \frac{c_i}{n} \geqslant S^l_i(E,c)$. By definition $IE_i(E,c) = IE_{i-1}(E,c) + \frac{c_i - c_{i-1}}{n-(i-1)} \geqslant \frac{c_i}{n}$. If IE satisfies RB, then $IE_{i-1}(E,c)$ should be $\geqslant \frac{c_{i-1}}{n}$. It easy to see that if this hold, then $IE_I(E,c) \geqslant \frac{c_i}{n} \geqslant S^l_i(E,c)$. Hence by replicating the same reasoning backwards, $IE_2(E,c) = IE_1(E,c) + \frac{c_2 - c_1}{n-1} \geqslant \frac{c_2}{n}$, and $IE_1(E,c) \geqslant \frac{c_1}{n}$, which always hold by the defintiont of the IE proposal. So, consider that $IE_1 = \frac{c_1}{n}$, then, $\frac{c_2 - c_1}{n-1} \geqslant \frac{c_2 - c_1}{n}$, and $IE_2(E,c) \geqslant S^l_2(E,c)$. Therefore, by backwards induction, the IE rule satisfies RB.

In order to prove that IE satisfies CGS and PRI, by definition, if $c_{i+1} > E > c_i$, for each $j \leqslant i$, $IE_j(E,c) = \frac{c_1}{n} + \sum\limits_{j=2}^{i} \frac{c_j - c_{j-1}}{n-j+1}$, and $S^l_j(E,c) = \frac{c_j}{n}$; Furthermore, for each $k \geqslant i+1$, $IE_k(E,c) = \frac{c_1}{n} + \sum\limits_{j=2}^{i} \frac{c_j - c_{j-1}}{n-j+1} + \frac{E - \sum_{j=2}^{i} IE_j(E,c)}{n-k+1}$, and $S^l_k(E,c) = \frac{E}{n}$.

Now consider that the endowment increases, $E' = c_{i+1}$, then for each $j \leqslant i$, $IE_j(E',c) = IE_j(E,c)$, and $S^l_j(E',c) = S^l_j(E,c)$. Furthermore, for each $k \geqslant i+1$, $IE_k(E',c) = \frac{c_1}{n} + \sum\limits_{j=2}^{i} \frac{c_j - c_{j-1}}{n-j+1} + \frac{c_{i+1} - c_i}{n-i+1} > IE_k(E,c)$, and $S^l_k(E',c) = \frac{c_i + 1}{n} > S^l_k(E,c) = \frac{E}{n}$. Hence, it is clear enough that only those agents whose securement increases, increase their allocation. Therefore, the IE satisfies CGS and PRI.

Finally, we are going to prove that, for each $(E,c) \in \mathcal{B}$ and each $i \in N$, a rule φ satisfying axioms $ETEB$, CGS and PRI coincides with the IE rule if $b(E,c) = S^l(E,c)$.

If $E \leqslant c_1, S^l_i(E,c) = \frac{E}{n}$, so by RB and efficiency, $\varphi_i(E,c) = \frac{E}{n} = IE_i(E,c)$.

If $c_1 < E' \leqslant c_2$, $S^l_1(E',c) = \frac{c_1}{n}$; and, for all $j \geqslant 2$, $S^l_j(E',c) = \frac{E'}{n}$. By RB, $\varphi_1(E',c) \geqslant \frac{c_1}{n}$, and $\varphi_j(E',c) \geqslant \frac{E'}{n}$. By PRI and CGS, only agents j, who have increased their lower bound, should be receive an equally increase of their allocation, i.e., $\varphi_1(E',c) = \frac{c_1}{n}$, and $\varphi_j(E',c) = \frac{c_1}{n} + \frac{E' - c_1}{n-1}$.

If $c_2 < E'' < c_3$, $S^l_1(E'',c) = \frac{c_1}{n}$, $S^l_2(E'',c) = \frac{c_2}{n}$, and $S^l_j(E'',c) = \frac{E''}{n}$, for each $j \geqslant 3$. By RB, $\varphi_1(E'',c) \geqslant \frac{c_1}{n}$, $\varphi_2(E'',c) \geqslant \frac{c_2}{n}$, and $\varphi_j(E',c) \geqslant \frac{E'}{n}$. By PRI and CGS, only agents j, who have increased their lower bound, should be receive an equally increase of their allocation, i.e., $\varphi_1(E',c) = \frac{c_1}{n}$, $\varphi_2(E'',c) = \frac{c_1}{n} + \frac{c_2 - c_1}{n-1}$, and $\varphi_j(E'',c) = \frac{c_1}{n} + \frac{c_2 - c_1}{n-1} + \frac{E'' - c_2}{n-2}$.

This argument is repeated until the endowment is greater than c_n, and we easily observe that the result is $\varphi(E, c) = IE(E, c)$. ■

4.4. ***Independence of the axioms***

The following examples provide the independence of the axioms used in some of the previous characterization results (we just do it with the fair lower bound).

Example 1. *Let $n = 3$ and the rule φ^a defined by:*

$$\varphi_i^a(E, (c_1, c_2, c_3)) = min\left\{c_i, \frac{E}{3}\right\}, \ \textit{for } i = 1, 2; \ \textit{and}$$

$$\varphi_3^a(E, (c_1, c_2, c_3)) = E - min\left\{c_1, \frac{E}{3}\right\} - min\left\{c_2, \frac{E}{3}\right\}.$$

It is clear that φ^a satisfies CRM and RB. But $\varphi^a(12, (1, 9, 10)) = (1, 4, 7)$, whereas the constrained equal awards rule is $cea(12, (1, 9, 10)) = (1, 5.5, 5.5)$. So φ^a does not satisfy ETEB, so neither does CGS.[1]

Example 2. *Let $n = 3$ and the rule φ^b defined by:*

$$\varphi^b(E, c) = cea(E, c), \ \textit{if } f_1^l(E, c) = f_2^l(E, c) = f_3^l(E, c);$$

$$\varphi^b(E, c) = cea(E, c) + (-x, -x, 2x), \ \textit{if } f_1^l(E, c) = f_2^l(E, c) < f_3^l(E, c); \ \textit{and}$$

$$\varphi^b(E, c) = cea(E, c) + (-2x, x, x), \ \textit{if } f_1^l(E, c) < f_2^l(E, c).$$

It is clear that ETEB and CGS are fulfilled. If we consider as in Example 1 the problem $(E, c) = (12, (1, 9, 10))$, then $\varphi^b(E, c) = (0, 6, 6)$, so RB and CRM are not satisfied.

Example 3. *Let $n = 3$ and consider the cea rule. It is clear enough that RB and CGS are fulfilled. If we consider the problems $(E, c) = (3, (3, 6, 9))$ and $(E', c) = (6, (3, 6, 9))$, then $cea(E, c) = (1, 1, 1)$, and $cea(E', c) = (2, 2, 2)$. Note that $s(E, c) = (1, 1, 1)$, and $s(E', c) = (1, 2, 2)$, so PRI is not satisfied.*

[1] In fact, $f^l(E, c) = (1, 4, 4)$, so *RB* implies that the second and third agent get the same amount, which is not the case with this rule.

5. Final Remarks

5.1. Extending the obtained results

We want to emphasize that, as shown, the claims rule obtained by asking some natural axioms crucially depends on the selected lower bound. We are now working with the *minimal right* lower bound in order to analyze the claims rule it defines when the *equal treatment* axiom is also required. We are also working in combining our axioms in the case of the *securement* lower bound in order to observe if we always retrieve the Ibn Ezra rule.

5.2. Duality

An important tool in the analysis of claims problems is the notion of duality: what happens if we observe losings instead of awards? We know that some rules are self-dual (distributing losses gives the same result as distributing award). But this is not true in general. Then, the dual of a claims rule φ its dual rule, φ^d, assigns losses in the same way as φ assigns gains (Aumann and Maschler, 1985), that is

$$\varphi_i^d(E, c) = c_i - \varphi_i(L, c).$$

We know that the *constrained equal awards* and the *constrained equal losses* rules are dual of each other (Herrero, 2003). On the other hand, two properties are dual if whenever a rule satisfies one of them, its dual rule satisfies the other.

The *fair* lower bound is defined by focusing on what claimants receive. By switching attention to the losses they incur, lower bounds on losses become upper bounds on awards.

An upper bound is a function, $d : \mathcal{B} \to \mathbb{R}^n_+$ such that for each $(E, c) \in \mathcal{B}$ and each $i \in N$, $d_i(E, c)$ is the maximal amount that agent i should receive in the claims problem (E, c) according to d.

Fair upper bound, (Moulin, 2002) f^u: for each $(E, c) \in \mathcal{B}$ and each $i \in N$, $f_i^u(E, c) = \max\left\{0, c_i - \frac{L}{n}\right\}$.

Note that these bounds f^u and f^l are dual of each other (Moulin, 2002), that is, $f_i^u(E, c) = c_i - f_i^l(L, c)$. Then, when a rule φ fulfills some axiom with respect to the fair lower bound, its dual rule fulfills the dual axiom with respect to the fair upper bound. The following result is an immediate consequence of Theorems 1 to 4 and duality.

Theorem 10. *Let us consider the bound $b = f^u$. Then, the cel rule is the only one satisfying:*

1. *RB^d and $ETEB^d$.*
2. *$ETEB^d$ and CRM^d.*
3. *RB^d and CGS^d.*
4. *CRM^d and CGS^d.*

The dual axiom of *ETEB* coincides with itself. The same occurs with *CGS*. On the other hand, the dual axioms of *RB* and *CRM* are obtained by reversing the inequality in the original properties.

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